15. Lists of singularities

In this Section the beginning of the hierarchy of classes of singularities of holomorphic functions is described.

15.0 Preliminary remarks

1. *Normal forms.* A *class of singularities* is a subset of the space of germs (or jets) of functions that is invariant under the action of the group of differmorphisms of the source preserving the origin. Orbits are examples of classes of singularities. Two germs (or jets) are said to be equivalent if they belong to the same orbit.

Another example of a class is the so-called $\mu = \text{const}$ *stratum*. The *multiplicity* (Milnor number) μ of a critical point $0 \in \mathbb{C}^n$ of a function f is the index of the singular point 0 of the vector field grad f. The $\mu = \text{const}$ *stratum for* f is defined as the connected component containing f of the space of germs with fixed multiplicity μ at 0.

To define a *normal form*, we regard the space of polynomials $M = \mathbb{C}[x_1, \ldots, x_n]$ as a subset of the space of germs of functions $f(x_1, \ldots, x_n)$ at 0.

A *normal form for a class K of functions* is given by a smooth map $\Phi\colon B \to M$ of a finite-dimensional linear *space of parameters* B into the space of polynomials for which the following three conditions hold:

(1) $\Phi(B)$ intersects all the orbits of K;

(2) the inverse image in B of each orbit is finite;

(3) the inverse image of the whole complement to K is contained in some proper hypersurface in B.

A normal form is said to be *polynomial* (respectively, *affine*) if the mapping Φ is polynomial (respectively, linear and inhomogeneous). An affine normal form is called *simple* if Φ has the form

$$\Phi(b_1, \ldots, b_r) = \varphi_0 + b_1\mathbf{x}^{\mathbf{m}_1} + \cdots + b_r\mathbf{x}^{\mathbf{m}_r},$$

where φ_0 is a fixed polynomial, the b_i are numbers and the $\mathbf{x}^{\mathbf{m}_i}$ are monomials. (In the applications the polynomial φ_0 is usually "simple" itself, that is, a sum of a few monomials.)

The existence of a unique normal form (at least a polynomial one) for the

whole stratum $\mu = \text{const}$ is by no means obvious a priori. An astonishing conclusion from our calculations is the existence of such normal forms for all the singularities of our list (therefore, in particular, for all singularities with one and two moduli). The majority of our forms are simple; it is probable that all the singularities in Chapter 15 have simple normal forms. We do not know how extensive the class of functions is for which the stratum $\mu = \text{const}$ admits a simple (or at least polynomial) normal form (this question is natural for the stable equivalence classes). J. Wahl and V. A. Vasil'ev [178] have given an example of a $\mu = \text{const}$ stratum not admitting an affine normal form: to it belongs

$$f = x^2y^2(x+y)^2(x+2y)^2 + x^9 - y^9.$$

2. *Series of singularities.* In the list the singularities are split into *series*, denoted by capital letters (we use light-face letters $A, \ldots, Z$ with various suffixes to denote the strata $\mu = \text{const}$ and bold-face letters $\boldsymbol{A}, \ldots, \boldsymbol{Z}$ with or without suffixes to denote classes of singularities that are unions of $\mu = \text{const}$ strata). Although the series undoubtedly exist, it is not at all clear what a series of singularities is.

For example, consider the series $\boldsymbol{A}$ and $\boldsymbol{D}$, which are formed by the orbits of the germs A_k: $f(x, y) = x^{k+1} + y^2$ and D_k: $f(x, y) = x^2y + y^{k-1}$. The classes A_k and D_k are adjacent to each other as follows*:

$$\begin{array}{ccccccccc} A_1 & \leftarrow & A_2 & \leftarrow & A_3 & \leftarrow & A_4 & \leftarrow & \ldots \\ & & & & \uparrow & & \uparrow & & \\ & & & & D_4 & \leftarrow & D_5 & \leftarrow & \ldots \end{array}$$

It is clear that there are two series in this example, $\boldsymbol{A}$ and $\boldsymbol{D}$. However, what is the formal meaning of this assertion, and does it exceed the limits of the arbitrary names?

Thus, to define the series $\boldsymbol{A}$ is to learn how to invert the arrows of the adjacencies, so that we go from A_k to A_{k+1}, without deviating through D_{k+1}. In the given case this is not hard to do (the singularities of $\boldsymbol{A}$ have a second differential of corank $\leqq 1$). In more complicated cases we can also state a rule for inverting the arrows (in each case separately). As a result there arise series with one or several suffixes (for example, the three-suffix series $T_{k,l,m} = axyz + x^k + y^l + z^m$), and the functions of the series may depend on parameters.

As in the example of the series $\boldsymbol{A}$ and $\boldsymbol{D}$ above, so in all cases, after a series

* A class of singularities L *is adjacent* to a class K (notation: $K \leftarrow L$) if every function $f \in L$ can be deformed into a function of K by an arbitrarily small perturbation.

is found, we can define it. However, a general definition of a series of singularities is not known. It is only clear that the series are associated with singularities of infinite multiplicity (for example, $D \sim x^2y$, $T \sim xyz$), so that the hierarchy of series reflects the hierarchy of non-isolated singularities.

3. *Periodicity.* The partition of many classes of singularities into $\mu =$ const strata reveals a peculiar periodicity, which can be described as follows. The whole stratification (partition) is a chain of identical fragments (beasts). Each beast consists of points (strata), two of which (the head and the tail) are distinguished. Apart from the head and tail, the beast can contain the strata joining them by arrows of adjacency, and extremities (series of infinite length). The head of each beast adheres to the tail of the preceding beast.

For example, the stratification of singularities of corank 2 with 3-jet x^3 is given by a chain of beasts, each of which consists of 5 points and one infinite leg:

$$\begin{array}{l} \bigcirc \leftarrow \cdot \leftarrow \cdot \leftarrow \cdot \leftarrow \bigcirc \\ \quad\;\; \uparrow \\ \quad\;\; \cdot \leftarrow \cdot \ldots \end{array}$$

(J_k in the list).

The reason for periodicity in the general case is not clear. A partial explanation has only been obtained for quasihomogeneous singularities by means of a technique resembling the fans of Enriques and Demazure (see [50]).

However, the periodicity becomes apparent not only under reduction of quasi-homogeneous functions to normal forms, but also in all the calculations associated with the classifications (so that for all the calculations it suffices, in fact, to consider only one beast from a chain).

As with the existence of series periodicity leads to the thought that the set of strata has some sort of algebraic structure.

D. Siersma [156] has shown a connection between periodicity and the resolution of singularities: translation by a period corresponds to a σ-process. Unfortunately it has not been possible from this remark to deduce the periodicity in the computations referred to above.

4. *Classes of small modality.* From the point of view of applications the most important characteristic of a class of singularities is its *codimension c* in the space of germs of functions with critical point O and critical value 0. A generic function has only singularities of codimension $c = 0$ (nondegenerate). Degenerate singularities are irremovable only in the case of a *family* of functions depending on parameters. In this case the class of codimension c is irremovable by a small perturbation only if the number of parameters is $l \geqq c$.

Thus, *in applications it is always necessary to study all classes up to codimension l* (that is, classes for which the complement to their union has codimension greater than l). This problem should not be confused with that of classifying singularities with orbital codimension $\leqq l$ (that is, with $\mu \leqq l + 1$). The latter problem occurs in applications only as a tool for solving the former.

From the topological point of view the most important characteristic of a singularity is the multiplicity μ of a critical point (equal to the number of simple critical points into which a composite point decomposes under a small perturbation).

The unexpected conclusion from these calculations is that *the algebraically most natural results are obtained not for the classification of classes of singularities up to a definite codimension c or multiplicity μ, but for the classification of classes of singularities of small modality m.*

The modality m is equal to the dimension of the stratum $\mu =$ const in the base of a versal deformation minus 1 (see [68]). Therefore, the codimension c of the stratum $\mu =$ const in the space of germs of functions with critical point O and critical value 0, the multiplicity μ and the modality m are connected by the relation

$$\mu = c + m + 1.$$

At the present time the following are completely classified:

(1) all the singularities for which $c \leqq 10$;
(2) all the singularities for which $\mu \leqq 16$;
(3) all the singularities for which $m \leqq 2$;

Singularities with $m = 0$, 1 and 2 are called, respectively, *simple*, *unimodal*, and *bimodal*. Lists of them are given below. An analysis of these lists shows that

(1) simple singularities are classified precisely by the Coxeter groups A_k, D_k, E_6, E_7, E_8 (that is, by the regular polyhedra in 3-space);
(2) unimodal singularities form a single infinite three-suffix series and 14 "exceptional" one-parameter families generated by quasihomogeneous singularities.

The quasihomogeneous unimodal singularities are obtained from automorphic functions connected with 14 distinguished triangles on the Lobachevskii plane and three distinguished triangles on the Euclidean plane in precisely the same way as simple singularities are connected with regular polyhedra (see [52]–[54]).

(3) Bimodal singularities form 8 infinite series and 14 exceptional two-parameter families generated by quasihomogeneous singularities.

The quasihomogeneous bimodal singularities are associated with the 6 quadrilaterals and the 14 triangles on the Lobachevskii plane (in the latter case one must consider automorphic functions with automorphy factors corresponding to 2-, 3- or 5-sheeted coverings) (see [53]).

All singularities with one and two moduli are classified precisely by the degenerations of elliptic curves, which were classified by Kodaira (see [103]). V. S. Kulikov has shown that to obtain these singularities from the degeneracies of elliptic curves it is necessary to blow up 1, 2 or 3 points in a minimal resolution of the degenerate fibre, after which the original fibre is blown down (see [103]).

Unfortunately, all the results listed here are obtained by a comparison of independently proved classification theorems, none of which can (so far) be derived from another.

15.1 Singularities with the number of moduli $m = 0$, 1 *and* 2

0. *Simple singularities* ($m = 0$). There are two infinite series A, D, and three exceptional singularities E_6, E_7, E_8:

A_k, $k \geq 1$	D_k, $k \geq 4$	E_6	E_7	E_8
x^{k+1}	$x^2y + y^{k-1}$	$x^3 + y^4$	$x^3 + xy^3$	$x^3 + y^5$

1. *Unimodal singularities* ($m = 1$). There are 3 families of parabolic singularities, a three-suffix series of hyperbolic singularities and 14 families of exceptional singularities.

Parabolic:

$\tilde{E}_6$	P_8	$x^3 + y^3 + z^3 + axyz$	$a^3 + 27 \neq 0$
$\tilde{E}_7$	X_9	$x^4 + y^4 + ax^2y^2$	$a^2 \neq 4$
$\tilde{E}_8$	J_{10}	$x^3 + y^6 + ax^2y^2$	$4a^3 + 27 \neq 0$

Hyperbolic:

$T_{p,q,r}$: $x^p + y^q + z^r + axyz$, $a \neq 0$, $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$.

14 Exceptional families:

E_{12}	$x^3 + y^7 + axy^5$	E_{13}	$x^3 + xy^5 + ay^8$
E_{14}	$x^3 + y^8 + axy^6$	Z_{11}	$x^3y + y^5 + axy^4$
Z_{12}	$x^3y + xy^4 + ax^2y^3$	Z_{13}	$x^3y + y^6 + axy^5$
W_{12}	$x^4 + y^5 + ax^2y^3$	W_{13}	$x^4 + xy^4 + ay^6$
Q_{10}	$x^3 + y^4 + yz^2 + axy^3$	Q_{11}	$x^3 + y^2z + xz^3 + az^5$
Q_{12}	$x^3 + y^5 + yz^2 + axy^4$	S_{11}	$x^4 + y^2z + xz^2 + ax^3z$
S_{12}	$x^2y + y^2z + xz^3 + az^5$	U_{12}	$x^3 + y^3 + z^4 + axyz^2$

2. *Bimodal singularities* ($m = 2$). There are 8 infinite series and 14 exceptional families. Let $a = a_0 + a_1 y$.

4 Infinite series of bimodal singularities of corank 2:

Notation	Normal form	Restrictions	Multiplicity μ
$J_{3,0}$	$x^3 + bx^2y^3 + y^9 + cxy^7$	$4b^3 + 27 \neq 0$	16
$J_{3,p}$	$x^3 + x^2y^3 + ay^{9+p}$	$p > 0, a_0 \neq 0$	$16 + p$
$Z_{1,0}$	$x^3y + dx^2y^3 + cxy^6 + y^7$	$4d^3 + 27 \neq 0$	15
$Z_{1,p}$	$x^3y + x^2y^3 + ay^{7+p}$	$p > 0$, $a_0 \neq 0$	$15 + p$
$W_{1,0}$	$x^4 + ax^2y^3 + y^6$	$a_0^2 \neq 4$	15
$W_{1,p}$	$x^4 + x^2y^3 + ay^{6+p}$	$p > 0$, $a_0 \neq 0$	$15 + p$
$W^{\sharp}_{1,2q-1}$	$(x^2 + y^3)^2 + axy^{4+q}$	$q > 0$, $a_0 \neq 0$	$15 + 2q - 1$
$W^{\sharp}_{1,2q}$	$(x^2 + y^3)^2 + ax^2y^{3+q}$	$q > 0$, $a_0 \neq 0$	$15 + 2q$

4 Infinite series of bimodal singularities of corank 3:

Notation	Normal form	Restrictions	Multiplicity μ
$Q_{2,0}$	$x^3 + yz^2 + ax^2y^2 + xy^4$	$a_0^2 \neq 4$	14
$Q_{2,p}$	$x^3 + yz^2 + x^2y^2 + ay^{6+p}$	$p > 0$, $a_0 \neq 0$	$14 + p$
$S_{1,0}$	$x^2z + yz^2 + y^5 + azy^3$	$a_0^2 \neq 4$	14
$S_{1,p}$	$x^2z + yz^2 + x^2y^2 + ay^{5+p}$	$p > 0$, $a_0 \neq 0$	$14 + p$
$S^{\sharp}_{1,2q-1}$	$x^2z + yz^2 + zy^3 + axy^{3+q}$	$q > 0$, $a_0 \neq 0$	$14 + 2q - 1$
$S^{\sharp}_{1,2q}$	$x^2z + yz^2 + zy^3$ $+ ax^2y^{2+q}$	$q > 0$, $a_0 \neq 0$	$14 + 2q$
$U_{1,0}$	$x^3 + xz^2 + xy^3 + ay^3z$	$a_0(a_0^2 + 1) \neq 0$	14
$U_{1,2q-1}$	$x^3 + xz^2 + xy^3 + ay^{1+q}z^2$	$q > 0$, $a_0 \neq 0$	$14 + 2q - 1$
$U_{1,2q}$	$x^3 + xz^2 + xy^3 + ay^{3+q}z$	$q > 0$, $a_0 \neq 0$	$14 + 2q$

14 Exceptional families

E_{18}	$x^3+y^{10}+axy^7$	E_{19}	$x^3+xy^7+ay^{11}$
E_{20}	$x^3+y^{11}+axy^8$	Z_{17}	$x^3y+y^8+axy^6$
Z_{18}	$x^3y+xy^6+ay^9$	Z_{19}	$x^3y+y^9+axy^7$
W_{17}	$x^4+xy^5+ay^7$	W_{18}	$x^4+y^7+ax^2y^4$
Q_{16}	$x^3+yz^2+y^7+axy^5$	Q_{17}	$x^3+yz^2+xy^5+ay^8$
Q_{18}	$x^3+yz^2+y^8+axy^6$	S_{16}	$x^2z+yz^2+xy^4+ay^6$
S_{17}	$x^2z+yz^2+y^6+azy^4$	U_{16}	$x^3+xz^2+y^5+ax^2y^2$

All the functions of these families (under the restrictions indicated in the tables) are bimodal.

15.2 Singularities of corank 2 with nonzero 4-jet

Throughout this section $a = a_0 + \ldots + a_{k-2}y^{k-2}(a = 0$ for $k = 1)$.

1. *Singularities of corank* 2 *with nonzero* 3*-jet*. Besides the simple singularities A, D, E_6, E_7, E_8 there is a further infinite series of classes:

Notation	Normal form	Restrictions	Multiplicity μ	Modality m
$J_{k,0}$	$x^3+bx^2y^k+y^{3k}+cxy^{2k+1}$	$k>1,\ 4b^3+27\neq 0$	$6k-2$	$k-1$
$J_{k,i}$	$x^3+x^2y^k+ax^{3k+i}$	$k>1,\ i>0,\ a_0\neq 0$	$6k-2+i$	$k-1$
E_{6k}	$x^3+y^{3k+1}+axy^{2k+1}$	$k\geqq 1$	$6k$	$k-1$
E_{6k+1}	$x^3+xy^{2k+1}+ay^{3k+2}$	$k\geqq 1$	$6k+1$	$k-1$
E_{6k+2}	$x^3+y^{3k+2}+axy^{2k+2}$	$k\geqq 1$	$6k+2$	$k-1$

Here $c = c_0 + \ldots + c_{k-3}y^{k-3}$ for $k > 2$; for $k = 2$ we set $c = 0$.

2. *Singularities of corank* 2 *with zero* 3*-jet and nonzero* 4*-jet*. There are 4 infinite series of classes X, Y, Z and W.

Singularities of classes X and Y:

Notation	Normal form $(k>1)$	Restrictions	Multiplicity μ	Modality m
$X_{k,0}$	$x^4+bx^3y^k+ax^2y^{2k}+xy^{3k}$	$\Delta\neq 0,\ a_0b_0\neq 9$	$12k-3$	$3k-2$
$X_{k,p}$	$x^4+ax^3y^k+x^2y^{2k}+by^{4k+p}$	$a_0^2\neq 4, b_0\neq 0, p>0$	$12k-3+p$	$3k-2$
$Y^k_{r,s}$	$[(x+ay^k)^2+by^{2k+s}](x^2+y^{2k+r})$	$1\leqq s\leqq r, a_0\neq 0\neq b_0$	$12k-3+r+s$	$3k-2$

In the case $k = 1$ these formulae need some modification:

$X_{1,0}$	$x^4+a_0x^2y^2+y^4$	$a_0^2\neq 4$	9	1
$X_{1,p}$	$x^4+x^2y^2+a_0y^{4+p}$	$a_0\neq 0$	$9+p$	1
$Y^1_{r,s}$	$x^{4+r}+a_0x^2y^2+y^{4+s}$	$a_0\neq 0$	$9+r+s$	1

Finally, $X_{1,0} = X_9$, $X_{1,p} = T_{2,4,4+p}$, $Y^1_{r,s} = T_{2,4+r,4+s}$ (see 15.1). Here

$$\Delta = 4(a_0^3 + b_0^3) - a_0^2b_0^2 - 18a_0b_0 + 27, \quad b = b_0 + \ldots + b_{2k-2}y^{2k-2}.$$

Singularities of class Z:
The singularities $Z^k_{i,0}$ and Z^k_{μ} $(k > 1)$ have a normal form of the kind $f = (x + ay^k)f_2$, where $a_0 \neq 0$ and f_2 is given by the following table:

Notation	f_2	Restrictions $(k>1)$	Multiplicity μ	Modality m
$Z^k_{i,0}$	$x^3+dx^2y^{k+i}+cxy^{2k+2i+1}+y^{3k+3i}$	$4d^3+27\neq 0,\ i>0$	$12k+6i-3$	$3k+i-2$
$Z^k_{12k+6i-1}$	$x^3+bxy^{2k+2i+1}+y^{3k+3i+1}$	$i\geqq 0$	$12k+6i-1$	$3k+i-2$
Z^k_{12k+6i}	$x^3+xy^{2k+2i+1}+by^{3k+3i+2}$	$i\geqq 0$	$12k+6i$	$3k+i-2$
$Z^k_{12k+6i+1}$	$x^3+bxy^{2k+2i+2}+y^{3k+3i+2}$	$i\geqq 0$	$12k+6i+1$	$3k+i-2$

The singularity $Z^k_{i,p}$ $(k > 1,\ i > 0,\ p > 0)$ has the normal form

$$Z^k_{i,p}\colon (x^2 + axy^k + by^{2k+i})(x^2 + y^{2k+2i+p}), \quad a_0 \neq 0, \quad b_0 \neq 0;$$

its multiplicity is $\mu = 12k + 6i + p - 3$, its modality is $m = 3k + i - 2$.
For $k = 1$ the preceding formulae are modified in the following way:
(1) the upper index k is dropped;
(2) the singularities $Z_{i,0}$, Z_{6i+11}, Z_{6i+12}, Z_{6i+13} $(i > 0)$ have normal forms of the kind $f = yf_2$, where f_2 is given by the preceding table;
(3) $Z_{i,p}\colon y(x^3 + x^2y^{i+1} + by^{3i+p+3})$, $b_0 \neq 0, i > 0, p > 0; \mu = 9 + 6i + p, m = i + 1$.
Throughout the formulae above we have used

$$b = b_0 + \ldots + b_{2k+i-2}y^{2k+i-2}, \quad c = c_0 + \ldots + c_{2k+i-3}y^{2k+i-3}.$$

Singularities of class W:

Notation	Normal form	Restrictions	Multiplicity μ	Modality m
W_{12k}	$x^4+y^{4k+1}+axy^{3k+1}+cx^2y^{2k+1}$	$k\geqq 1$	$12k$	$3k-2$
W_{12k+1}	$x^4+xy^{3k+1}+ax^2y^{2k+1}+cy^{4k+2}$	$k\geqq 1$	$12k+1$	$3k-2$
$W_{k,0}$	$x^4+bx^2y^{2k+1}+axy^{3k+2}+y^{4k+2}$	$k\geqq 1$, $b_0^2\neq 4$	$12k+3$	$3k-1$
$W_{k,i}$	$x^4+ax^3y^{k+1}+x^2y^{2k+1}+by^{4k+2+i}$	$i>0$, $b_0\neq 0$	$12k+3+i$	$3k-1$
$W^{\#}_{k,2q-1}$	$(x^2+y^{2k+1})^2+bxy^{3k+1+q}+ay^{4k+2+q}$	$q>0$, $b_0\neq 0$	$12k+2+2q$	$3k-1$
$W^{\#}_{k,2q}$	$(x^2+y^{2k+1})^2+bx^2y^{2k+1+q}+axy^{3k+2+q}$	$q>0$, $b_0\neq 0$	$12k+3+2q$	$3k-1$
W_{12k+5}	$x^4+xy^{3k+2}+ax^2y^{2k+2}+by^{4k+3}$	$k\geqq 1$	$12k+5$	$3k-1$
W_{12k+6}	$x^4+y^{4k+3}+axy^{3k+3}+bx^2y^{2k+2}$	$k\geqq 1$	$12k+6$	$3k-1$

In these formulae $b=b_0+\ldots+b_{2k-1}y^{2k-1}$, $c=c_0+\ldots+c_{2k-2}y^{2k-2}$; as elsewhere, $a=a_0+\ldots+a_{k-2}y^{k-2}$ for $k>1$ and $a=0$ for $k=1$.

15.3 Singularities of corank 3 with a reduced 3-jet

Besides the unimodal singularities of series T (see Section 15.1) there are 3 infinite series of classes of such singularities: $\boldsymbol{Q}$, $\boldsymbol{S}$ and $\boldsymbol{U}$.

1. *Series* $\boldsymbol{Q}$. Singularities with 3-jet x^3+yz^2 form a single infinite series of classes

Notation	Normal form	Restrictions	Multiplicity μ	Modality m
$Q_{k,0}$	$\varphi+bx^2y^k+xy^{2k}$	$k>1$, $b_0^2\neq 4$	$6k+2$	k
$Q_{k,i}$	$\varphi+x^2y^k+by^{3k+i}$	$k>1$, $b_0\neq 0$	$6k+2+i$	k
Q_{6k+4}	$\varphi+y^{3k+1}+bxy^{2k+1}$	$k\geqq 1$	$6k+4$	k
Q_{6k+5}	$\varphi+xy^{2k+1}+by^{3k+2}$	$k\geqq 1$	$6k+5$	k
Q_{6k+6}	$\varphi+y^{3k+2}+bxy^{2k+2}$	$k\geqq 1$	$6k+6$	k

In these formulae $\varphi=x^3+yz^2$, $b=b_0+\ldots+b_{k-1}y^{k-1}$.

2. *Series* $\boldsymbol{S}$. Singularities with 3-jet x^2z+yz^2 form a single infinite series of classes

Notation	Normal form	Restrictions	Multiplicity μ	Modality m
S_{12k-1}	$\varphi+y^{4k}+axy^{3k}+czy^{2k+1}$	—	$12k-1$	$3k-2$
S_{12k}	$\varphi+xy^{3k}+cy^{4k+1}+azy^{2k+1}$	—	$12k$	$3k-2$
$S_{k,0}$	$\varphi+y^{4k+1}+axy^{3k+1}+bzy^{2k+1}$	$b_0^2\neq 4$	$12k+2$	$3k-1$
$S_{k,i}$	$\varphi+x^2y^{2k}+ax^3y^k+by^{4k+1+i}$	$i>0$, $b_0\neq 0$	$12k+2+i$	$3k-1$
$S^{\#}_{k,2q-1}$	$\varphi+zy^{2k+1}+bxy^{3k+q}+ay^{4k+q+1}$	$q>0$, $b_0\neq 0$	$12k+2q+1$	$3k-1$
$S^{\#}_{k,2q}$	$\varphi+zy^{2k+1}+bx^2y^{2k+q}+axy^{3k+q+1}$	$q>0$, $b_0\neq 0$	$12k+2q+2$	$3k-1$
S_{12k+4}	$\varphi+xy^{3k+1}+azy^{2k+2}+by^{4k+2}$	—	$12k+4$	$3k-1$
S_{12k+5}	$\varphi+y^{4k+2}+axy^{3k+2}+bzy^{2k+2}$	—	$12k+5$	$3k-1$

In these formulae $\varphi=x^2z+yz^2$, $a=a_0+\ldots+a_{k-2}y^{k-2}$ for $k>1$, $a=0$ for $k=1$; $b=b_0+\ldots+b_{2k-1}y^{2k-1}$, $c=c_0+\ldots+c_{2k-2}y^{2k-2}$.

Besides these there are further classes $\boldsymbol{S}_k^{*}(k>1)$, subdivided into $S^{*}_{k,0}$, $\boldsymbol{SP}_k$, $\boldsymbol{SQ}_k$, $\boldsymbol{SR}_k$, where $\mu(S^{*}_{k,0})=12k-4$, $m(S^{*}_{k,0})=\ldots=m(\boldsymbol{SR}_k)=3k-2$, codim $S_k^{*}=9k-3$.

3. *Series* $\boldsymbol{U}$. The singularities with 3-jet x^3+xz^2 form a single infinite series of classes

Notation	Normal form	Restrictions	Multiplicity μ	Modality m
U_{12k}	$\varphi+y^{3k+1}+axy^{2k+1}+bzy^{2k+1}+dx^2y^{k+1}$	—	$12k$	$4k-3$
$U_{k,2q}$	$\varphi+xy^{2k+1}+ax^2y^{k+1}+by^{3k+2+q}+czy^{2k+1+q}$	$q\geqq 0$, $c_0\neq 0$	$12k+2+2q$	$4k-2$
$U_{k,2q-1}$	$\varphi+xy^{2k+1}+ax^2y^{k+1}+bzy^{2k+1+q}+cz^2y^{k+q}$	$q>0$, $c_0\neq 0$	$12k+1+2q$	$4k-2$
U_{12k+4}	$\varphi+y^{3k+2}+axy^{2k+2}+bzy^{2k+2}+cx^2y^{k+1}$	—	$12k+4$	$4k-2$

In these formulae $\varphi=x^3+xz^2$; $c_0^2+1\neq 0$ for $q=0$; everywhere,

$$a=a_0+\ldots+a_{k-2}y^{k-2}\quad\text{for}\quad k>1,\quad a=0\quad\text{for}\quad k=1;$$

$$b=b_0+\ldots+b_{k-2}y^{k-2}\quad\text{for}\quad k>1,\quad b=0\quad\text{for}\quad k=1;$$

$$c=c_0+\ldots+c_{2k-1}y^{2k-1},\quad d=d_0+\ldots+d_{2k-2}y^{2k-2}.$$

Besides this there are the further classes U^*_k $(k > 1)$, subdivided into classes $U^*_{k,0}$, UP_k, UQ_k, UR_k, US_k, UT_k, for which $\mu \geqq \mu(U^*_{k,0}) = 12k - 4$, $m(U^*_{k,0}) = \ldots = m(UT_k) = 4k - 3$, codim $(U^*_k) = 8k - 2$.

15.4 Series V

The singularities of corank 3 with 3-jet x^2y are subdivided into classes $V_{1,i}$, $V^\sharp_{1,i}$, V^*, where

Notation	Normal form	Restrictions	Multiplicity μ	Modality m
$V_{1,0}$	$x^2y + z^4 + az^3y + bz^2y^2 + zy^3$	$\Delta(a, b_0) \neq 0$	15	3
$V_{1,p}$	$x^2y + z^4 + bz^3y + z^2y^2 + ay^{4+p}$	$b^2 \neq 4$, $a_0 \neq 0$	$15 + p$	3
$V^\sharp_{1,2q-1}$	$x^2y + z^3y + ay^2z^2 + y^4 + bxz^{2+q}$	$4a^3 + 27 \neq 0$, $b_0 \neq 0$	$14 + 2q$	3
$V^\sharp_{1,2q}$	$x^2y + z^3y + ay^2z^2 + y^4 + bz^{4+q}$	$4a^3 + 27 \neq 0$, $b_0 \neq 0$	$15 + 2q$	3

Here $p > 0$, $q > 0$, $a = a_0 + a_1y$, $b = b_0 + b_1z$. The singularities of class V^* satisfy the conditions:

$$\mu(V^*) \geqq 17, \quad m(V^*) \geqq 3, \quad \operatorname{codim}(V^*) = 13.$$

15.5 Other singularities

All the singularities whose normal forms are not given belong to the following 7 classes:

Notation	Corank	Adjacencies	Determination	$c \geqq$	$\mu \geqq$	$m \geqq$	Theorem
N	2	$N \to W_{13}$	$j_4 = 0$	12	16	3	47–49
S^*	3	$S^*_k \to S_{12k-7}$	see Theorem 77	15	20	4	77–81
U^*	3	$U^*_k \to U_{12k-8}$	see Theorem 90	14	20	5	90–96
V^*	3	$V^* \to V_{1,1} \cup V^\sharp_{1,1}$	see Theorem 98	13	17	3	97–102
V'	3	$V' \to V$	$j_3 = x^3$	13	18	4	103
V''	3	$V'' \to V'$	$j_3 = 0$	16	27	10	104
O	>3	$O \to T_{4,4,4}$	corank $\geqq 4$	10	16	5	105

Here $k \geqq 2$. The theorem numbers refer to the theorems of Chapter 16. The normal form for singularities of class O, excluding a set of codimension $c = 11$, is as follows:

$$x^3 + y^3 + z^3 + u^3 + (ax + by + cz + du)^3 + exyzu, \quad \Delta(a, b, c, d) \neq 0,$$

where Δ is the discriminant.

15.6 Some adjacencies

Only those adjacencies are shown below which arise naturally in the course of the classification (see Chapter 16).*

1. *Singularities with number of moduli $m = 0$, 1 or 2*

1.0. *Some adjacencies of the simple singularities.*

$$\begin{tikzcd}
A & D \arrow[l] & E_6 \arrow[l] & E_7 \arrow[l] & E_8, \arrow[l] \\
 & & (P) \arrow[u] & (X) \arrow[u] & (J) \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
A = A_1 & A_2 \arrow[l] & A_3 \arrow[l] & A_4 \arrow[l] & \ldots \arrow[l] \\
 & & & (D) \arrow[u] & \\
D = D_4 & D_5 \arrow[l] & D_6 \arrow[l] & \ldots \arrow[l] & \\
 & (E) \arrow[u] & & &
\end{tikzcd}$$

The classes P, X, J consist of non-simple singularities. All the decompositions of the simple singularities were described by O. V. Lyashko in [117].

1.1 *Some adjacencies of the unimodal singularities*

$$\begin{tikzcd}
 & J_{10} = T_{2,3,6} \arrow[r] & (E_8) & & & \\
\cdots \arrow[r] & T_{2,3,8} \arrow[r] & T_{2,3,7} \arrow[ul] & E_{12} \arrow[l] & E_{13} \arrow[l] & E_{14} \arrow[l] & (J_3) \arrow[l]
\end{tikzcd}$$

$$\begin{tikzcd}
 & & X_9 = T_{2,4,4} \arrow[r] & (E_7) & & \\
\cdots \arrow[r] & T_{2,4,6} \arrow[r] & T_{2,4,5} \arrow[u] & Z_{11} \arrow[l] & Z_{12} \arrow[l] & Z_{13} \arrow[l] & (Z^1_1) \arrow[l] \\
::: \arrow[r, shift left] \arrow[r, shift right] & T_{2,5,6} \arrow[r] \arrow[u] & T_{2,5,5} \arrow[u, "\geqq"'] & W_{12} \arrow[l] & W_{13} \arrow[l] \arrow[u] & (W_{1,0};\, N) \arrow[l] &
\end{tikzcd}$$

* These adjacencies possess the following property: an adjacency of the form $K \leftarrow L$ excludes the adjacency of any part K' of K to any part of L' of L (for example, $A \leftarrow D$ excludes $A_\mu \to D_4$). Such adjacencies are said to be *strong* adjacencies.

$$
\begin{tikzcd}[column sep=small, row sep=small]
 & & P_8 = T_{3,3,3} \arrow[r] & (E_6) & & & \\
\cdots \arrow[r] & T_{3,3,5} \arrow[r] & T_{3,3,4} \arrow[u] & Q_{10} \arrow[l] & Q_{11} \arrow[l] & Q_{12} \arrow[l] & (Q_2) \arrow[l] \\
\vdots\vdots\rightrightarrows & T_{3,4,5} \arrow[r] \arrow[u] & T_{3,4,4} \arrow[u] & S_{11} \arrow[l] \arrow[u] & S_{12} \arrow[l] \arrow[u] & (S_{1,0}) \arrow[l] & \\
\vdots\vdots\rightrightarrows & T_{4,4,5} \arrow[r] \arrow[u] & T_{4,4,4} \arrow[u] & U_{12} \arrow[l] \arrow[u] & U_{1,0};\ V) \arrow[l] & & \\
 & & (O) \arrow[u] & & & &
\end{tikzcd}
$$

The singularity classes in brackets are those of modality not equal to 1. A complete list of the adjacencies of all singularities of modality 1 was found by Brieskorn [42]. An analysis of the table of adjacencies indicates the semicontinuity of the spectrum of singularities (the set of Steenbrink indices $l_1 \leqq \ldots \leqq l_\mu$, see [159]). Brieskorn's list of the adjacencies of the unimodal singularities contains precisely all the adjacencies compatible with the semicontinuity hypothesis except one ($Q_{11} \to J_{10}$), V. V. Goryunov has verified that for all these singularities the hypothesis on the semicontinuity of the spectrum of the singularities holds (see [27]) and that the semicontinuity condition excludes the possibility of all adjacencies except those found by Brieskorn and except one other, the last adjacency being excluded by the semicontinuity of the indices of inertia of the intersection forms.

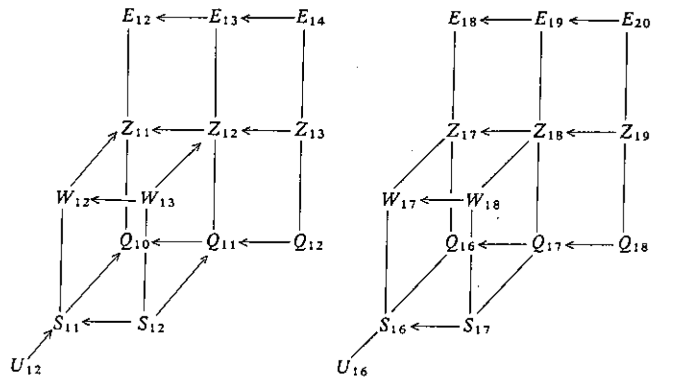
1.2. *Some adjacencies of bimodal singularities*

$$
\begin{tikzcd}[column sep=small, row sep=small]
 & \cdots \arrow[d] & & & \\
J_{3,0} & J_{3,1} \arrow[l] & E_{18} \arrow[l] & E_{19} \arrow[l] & E_{20} \arrow[l]
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small, row sep=small]
 & \cdots \arrow[d] & & & \\
Z_{1,0} & Z_{1,1} \arrow[l] & Z_{17} \arrow[l] & Z_{18} \arrow[l] & Z_{19} \arrow[l]
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=small, row sep=small]
 & W_{1,1} \arrow[dl] & \cdots \arrow[l] \\
W_{1,0} \arrow[dr] & & W_{17} \arrow[ul] \arrow[dl] & W_{18} \arrow[l] \\
 & W^{\#}_{1,1} & \cdots \arrow[l]
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small, row sep=small]
 & S_{1,1} \arrow[dl] & \cdots \arrow[l] \\
S_{1,0} & & S_{16} \arrow[ul] \arrow[dl] & S_{17} \arrow[l] \\
 & S^{\#}_{1,1} \arrow[ul] & \cdots \arrow[l]
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=small, row sep=small]
Q_{2,0} & Q_{2,1} \arrow[l] & Q_{16} \arrow[l] & Q_{17} \arrow[l] & Q_{18} \arrow[l] \\
 & \cdots \arrow[u] & & &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small, row sep=small]
U_{1,0} & U_{1,1} \arrow[l] & U_{16} \arrow[l] \\
 & \cdots \arrow[u] &
\end{tikzcd}
$$

The pyramids of exceptional singularities of modalities 1 and 2:



The vertical segments join singularities obtained from the same Kodaira class by means of the construction of V. S. Kulikov [103].

2. *Singularities of corank 2 with nonzero 4-jets*

2.1. *Singularities of corank 2 with nonzero 3-jets*

$$J = \leftarrow J_2 \leftarrow J_3 \leftarrow \ldots$$

$$
\begin{tikzcd}[column sep=small, row sep=small]
J_k = \ \leftarrow J_{k,0} & J_{k,1} \arrow[l] & E_{6k} \arrow[l] & E_{6k+1} \arrow[l] & E_{6k+2} \arrow[l] & (J_{k+1}) \arrow[l] \\
 & J_{k,0} \arrow[u] & J_{k,3} \arrow[l] & \ldots \arrow[l] & &
\end{tikzcd}
$$

2.2. *Singularities of corank 2 with zero 3-jet and nonzero 4-jet*
All such singularities form a single infinite series of classes

$$X = \leftarrow X_1 \leftarrow X_2 \leftarrow \ldots;$$

where

$$
\begin{tikzcd}[column sep=small, row sep=small]
 & & Y_k \arrow[dl] & \\
X_k = X_{k,0} & X_k^* \arrow[l] & & W_k \arrow[ul] \arrow[dl] & (X_{k+1}), \arrow[l] \\
 & & Z_k \arrow[ul] &
\end{tikzcd}
$$

and moreover

$$X_k^* = \leftarrow X_{k,1} \leftarrow X_{k,2} \leftarrow \ldots, \qquad Y_k = \leftarrow Y_{1,1}^k \leftarrow Y_{2,1}^k \leftleftarrows \ldots,$$

$$Z_k = \leftarrow Z_0^k \leftarrow Z_1^k \leftarrow \ldots, \qquad Z_0^k = \leftarrow Z_{12k-1}^k \leftarrow Z_{12k}^k \leftarrow Z_{12k+1}^k \leftarrow,$$

$$\begin{tikzcd} Z_i^k = \leftarrow Z_{i,0}^k \arrow[r, leftarrow] & Z_{i,1}^k \arrow[r, leftarrow] & Z_{12k+6i-1}^k \leftarrow Z_{12k+6i}^k \leftarrow Z_{12k+6i+1}^k \leftarrow (Z_{i+1}^k) \\ & Z_{i,2}^k \leftarrow Z_{i,3}^k \leftarrow \ldots (i > 0), \arrow[u] & \end{tikzcd}$$

$$\begin{tikzcd} & W_{k,1} \leftarrow W_{k,2} \leftarrow \ldots \arrow[dl] & \\ W_k = \leftarrow W_{12k} \leftarrow W_{12k+1} \leftarrow W_{k,0} & & W_{12k+5} \leftarrow W_{12k+6} \leftarrow (X_{k+1}) \arrow[ul] \arrow[dl] \\ & W_{k,1}^{\#} \leftarrow W_{k,2}^{\#} \leftarrow \ldots \arrow[ul] & \end{tikzcd}$$

3. *Singularities of corank 3 with reduced 3-jet.*

3.1. *Singularities of corank with 3-jet $x^3 + yz^2$*

$$Q = \leftarrow Q_1 \leftarrow Q_2 \leftarrow \ldots, \quad \textit{where} \quad Q_1 = \leftarrow Q_{10} \leftarrow Q_{11} \leftarrow Q_{12} \leftarrow,$$

$$\begin{tikzcd} Q_k = \leftarrow Q_{k,0} \arrow[r, leftarrow] & Q_{k,1} \leftarrow Q_{6k+4} \leftarrow Q_{6k+5} \leftarrow Q_{6k+6} \leftarrow (Q_{k+1}). \\ (k > 1) & Q_{k,2} \leftarrow Q_{k,3} \leftarrow \ldots \arrow[u] \end{tikzcd}$$

3.2. *Singularities of corank 3 with 3-jet $x^2z + yz^2$*

$$S = \leftarrow S_1 \leftarrow S_2 \leftarrow \ldots,$$

where

$$\begin{tikzcd} & S_{k,1} \leftarrow \ldots \arrow[dl] & \\ S_k = \leftarrow S_{12k-1} \leftarrow S_{12k} \leftarrow S_{k,0} & & S_{12k+4} \leftarrow S_{12k+5} \leftarrow S_{k+1}^* \leftarrow (S_{k+1}) \arrow[ul] \arrow[dl] \\ & S_{k,1}^{\#} \leftarrow \ldots \arrow[ul] & \end{tikzcd}$$

The singularities S_k^* $(k > 1)$ are subdivided into classes as follows:

$$\begin{tikzcd} & SQ_k \arrow[dl] & \\ S_k^* = S_{k,0}^* \leftarrow SP_k & & (S_k). \arrow[ul] \arrow[dl] \\ & SR_k \arrow[ul] & \end{tikzcd}$$

3.3. *Singularities of corank 3 with 3-jet $x^3 + xz^2$.*

$$U = \leftarrow U_1 \leftarrow U_2 \leftarrow \ldots,$$

where

$$\begin{tikzcd} U_k = \leftarrow U_{12k} \leftarrow U_{k,0} \arrow[r, leftarrow] & U_{k,1} \leftarrow U_{12k+4} \leftarrow U_{k+1}^* \leftarrow (U_{k+1}). \\ & U_{k,2} \leftarrow \ldots \arrow[u] \end{tikzcd}$$

$$\begin{tikzcd} U_k^* = \leftarrow U_{k,0}^* \arrow[r, leftarrow] & UP_k \arrow[r, leftarrow] & UR_k \arrow[r, leftarrow] & UT_k \\ & UQ_k \arrow[u] \arrow[r, leftarrow] & US_k \arrow[u] \arrow[r, leftarrow] & (U_k). \arrow[u] \end{tikzcd}$$

4. *Singularities of corank 3 with 3-jet x^2y*

$$\begin{tikzcd} & V_{1,1} \leftarrow V_{1,2} \leftarrow \ldots \arrow[dl] & \\ V = V_{1,0} & & V^* \arrow[ul] \arrow[dl] \\ & V_{1,1}^{\#} \leftarrow V_{1,2}^{\#} \leftarrow \ldots \arrow[ul] & \end{tikzcd}$$

16. The determinator of singularities

The 105 theorems given below allow one to find the place of any singularity in the lists of Chapter 15.

16.1 Notations

f is the germ of a holomorphic function at an *isolated* critical point O of multiplicity μ, or its Taylor series at O, or a formal series in the variables x, y or x, y, z. having finite μ.

$f \sim g$ means that the germs f and g at O are equivalent (that is, there exists a germ of a diffeomorphism or a formal series h such that $f = g \circ h$).

$\Rightarrow$ means "implies".

$\models\!\!\Rightarrow$ means "see" (references of the form $\models\!\!\Rightarrow i$ are not parts of the statements of the theorems; they indicate the number of the theorem where the singularities of that class are classified).

$j_k f$ means the k-jet of f at O (or the Taylor polynomial of order k at O).

$A, \ldots, Z$ are the stable equivalence classes of germs of functions that are defined in Chapter 15.

$m(f)$ is the modality of the germ f at O.

$c(f)$ is the codimension of the stratum $\mu = \text{const}$ of the germ of f in the space of germs of functions with critical point O and critical value 0.

$c(K)$ is the codimension of the class K in that space.

$j_{\{x^{m_i}\}} f$ is the quasijet of f at O, defined by the monomials x^{m_i} (or the corresponding Taylor polynomial).*

* A system of n monomials $\{x^{m_i}\}$ in $x_1, \ldots, x_n$ with independent exponents $m_i \in \mathbb{Z}^n \subset \mathbb{R}^n$ defines a hyperplane $\Gamma \subset \mathbb{R}^n$, $\Gamma = \{m : (\alpha, m) = 1\}$. If all the components α_i of the vector α are positive, then α is called the *quasihomogeneity type*, and the number $(\alpha, \mathbf{m})$ is the *degree* of the monomial x^m. The polynomial $f = \Sigma f_m x^m$ is quasihomogeneous of degree d and type α if $(\alpha, \mathbf{m}) = \forall dm : f_m \neq 0$.
The quasihomogeneity type defines a grading and a decreasing ring filtration $\mathscr{E}_0 \supset \ldots$, where

$$\mathscr{E}_d = \{f : (\alpha, \mathbf{m}) \geq \forall dm : f_m \neq 0\}.$$

The quotient-space $\mathscr{E}_0 / \cup \mathscr{E}_d$, $d > 1$, is called a *space of quasijets* defined by the monomials $\{x^{m_i}\}$ (or defined by the quasihomogeneity type α). For a fixed coordinate system quasijets can be identified with polynomials whose monomials are all of order at most 1 (that is, whose exponents lie on Γ or on the same side of Γ as O).
Quasihomogeneous diffeomorphisms are diffeomorphisms of $\mathbb{C}^n$ that preserve the grading of the ring $\mathbb{C}[[x_1, \ldots, x_n]]$. The Lie group of quasihomogeneous diffeomorphisms acts on the spaces of quasijets and on the spaces of quasihomogeneous polynomials. *Quasihomogeneous equivalence* refers to membership in a single orbit of this action.

$j_{\{x^{m_i}\}} f \approx g$ denotes quasihomogeneous equivalence of jets or Taylor polynomials.

j^*, φ – the meaning of these symbols for Theorems **58–65**, **66–81**, **82–89**, **98–102** is explained before the first theorem of each group.

Δ is the discriminant. In theorems **36**, **37**, **47**, **48**, **98**, and **99**, $\Delta = 4(a^3 + b^3) + 27 - a^2b^2 - 18ab$.

16.2 The determinator

1. $\mu(f) < \infty \Rightarrow$ one of four cases:

$$\begin{aligned} \operatorname{corank} f &\leq 1 \models\!\!\Rightarrow \mathbf{2}; \\ &= 2 \models\!\!\Rightarrow \mathbf{3}; \\ &= 3 \models\!\!\Rightarrow \mathbf{50}; \\ &> 3 \models\!\!\Rightarrow \mathbf{105}. \end{aligned}$$

2. $\operatorname{corank} f \leq 1 \Rightarrow f \in A_k$ $(k \geq 1)$.

In Theorems **3–49**, $f \in \mathbb{C}[[x, y]]$.

3. $j_2 f = 0 \Rightarrow$ one of four cases;

$$\begin{aligned} j_3 f &\approx x^2y + y^3 \models\!\!\Rightarrow \mathbf{4}; \\ &\approx x^2y \quad \models\!\!\Rightarrow \mathbf{5}; \\ &\approx x^3 \quad \models\!\!\Rightarrow \mathbf{6_1}; \\ &= 0 \quad \models\!\!\Rightarrow \mathbf{13}. \end{aligned}$$

4. $j_3 f = x^2y + y^3 \Rightarrow f \in D_4$.

5. $j_3 f = x^2y \Rightarrow f \in D_k$ $(k > 4)$.

In Theorems **6–9** the number $k \geq 1$.

6$_k$. $j_{x^3, y^{3k}} f(x, y) = x^3 \Rightarrow$ one of four cases:

$$\begin{aligned} j_{x^3, y^{3k+1}} f &\approx x^3 + y^{3k+1} \models\!\!\Rightarrow \mathbf{7_k}; \\ j_{x^3, xy^{2k+1}} f &\approx x^3 + xy^{2k+1} \models\!\!\Rightarrow \mathbf{8_k}; \\ j_{x^3, y^{3k+2}} f &\approx x^3 + y^{3k+2} \models\!\!\Rightarrow \mathbf{9_k}; \\ j_{x^3, y^{3k+2}} f &= x^3 \models\!\!\Rightarrow \mathbf{10_{k+1}}. \end{aligned}$$

7$_k$. $j_{x^3, y^{3k+1}} f = x^3 + y^{3k+1} \Rightarrow f \in E_{6k}$.

8$_k$. $j_{x^3, xy^{2k+1}} f = x^3 + xy^{2k+1} \Rightarrow f \in E_{6k+1}$.

9$_k$. $j_{x^3, y^{3k+2}} f = x^3 + y^{3k+2} \Rightarrow f \in E_{6k+2}$.

In Theorems **10–12**, $k > 1$.

10$_k$. $j_{x^3, y^{3k-1}} f = x^3 \Rightarrow$ one of three cases:

$$\begin{aligned} j_{x^3,y^{3k}}f &\approx x^3+ax^2y^k+y^{3k}, \quad 4a^3+27\neq 0 &&\models\!\Rightarrow \mathbf{11}_k;\\ &\approx x^3+x^2y^k &&\models\!\Rightarrow \mathbf{12}_k;\\ &\approx x^3 &&\models\!\Rightarrow \mathbf{6}_k. \end{aligned}$$

11$_k$. $j_{x^3,y^{3k}}f = x^3+ax^2y^k+y^{3k}, \quad 4a^3+27\neq 0 \Rightarrow f\in J_{k,0}$.
12$_k$. $j_{x^3,y^{3k}}f = x^3+x^2y^k \quad \Rightarrow f\in J_{k,p}\ (p>0)$.

The series X_1
13. $j_3f(x,y)=0 \Rightarrow$ one of six cases:

$$\begin{aligned} j_4f &\approx x^4+ax^2y^2+y^4, \quad a^2\neq 4 &&\models\!\Rightarrow \mathbf{14};\\ &\approx x^4+x^2y^2 &&\models\!\Rightarrow \mathbf{15};\\ &\approx x^2y^2 &&\models\!\Rightarrow \mathbf{16};\\ &\approx x^3y &&\models\!\Rightarrow \mathbf{17};\\ &\approx x^4 &&\models\!\Rightarrow \mathbf{25};\\ &= 0 &&\models\!\Rightarrow \mathbf{47}. \end{aligned}$$

14. $j_4f = x^4+ax^2y^2+y^4, \quad a^2\neq 4 \Rightarrow f\in X_9 = X_{1,0} = T_{2,4,4}$.
15. $j_4f = x^4+x^2y^2 \quad \Rightarrow f\in X_{1,p} = T_{2,4,4+p}\ (p>0)$.
16. $j_4f = x^2y^2 \quad \Rightarrow f\in Y^1_{p,q} = T_{2,4+p,4+q}\ (p\geqq q>0)$.
17. $j_4f = x^3y \quad \Rightarrow j_{x^3y,y^4}f = x^3y \Rightarrow \mathbf{18}_1$.

In Theorems **18–21**, $p\geqq 1$.
18$_p$. $j_{x^3y,y^{3p+1}}f = x^3y \Rightarrow$ one of four cases:

$$\begin{aligned} j_{x^3y,y^{3p+2}}f &\approx x^3y+y^{3p+2} &&\models\!\Rightarrow \mathbf{19}_p;\\ j_{x^3y,xy^{2p+2}}f &\approx x^3y+xy^{2p+2} &&\models\!\Rightarrow \mathbf{20}_p;\\ j_{x^3y,y^{3p+3}}f &\approx x^3y+y^{3p+3} &&\models\!\Rightarrow \mathbf{21}_p;\\ j_{x^3y,y^{3p+3}}f &= x^3y &&\models\!\Rightarrow \mathbf{22}_{p+1}. \end{aligned}$$

19$_p$. $j_{x^3y,y^{3p+2}}f = x^3y+y^{3p+2} \Rightarrow f\in Z_{6p+5}$.
20$_p$. $j_{x^3y,xy^{2p+2}}f = x^3y+xy^{2p+2} \Rightarrow f\in Z_{6p+6}$.
21$_p$. $j_{x^3y,y^{3p+3}}f = x^3y+y^{3p+3} \Rightarrow f\in Z_{6p+7}$.

In Theorems **22–24**, $p>1$.
22$_p$. $j_{x^3y,y^{3p}}f = x^3y \Rightarrow$ one of three cases:

$$\begin{aligned} j_{x^3y,y^{3p+1}}f &\approx y(x^3+bx^2y^p+y^{3p}), \quad 4b^3+27\neq 0 &&\models\!\Rightarrow \mathbf{23}_p;\\ &\approx y(x^3+x^2y^p) &&\models\!\Rightarrow \mathbf{24}_p;\\ &\approx x^3y &&\models\!\Rightarrow \mathbf{18}_p. \end{aligned}$$

23$_p$. $j_{x^3y,y^{3p+1}}f = y(x^3+bx^2y^p+y^{3p}), \quad 4b^3+27\neq 0 \Rightarrow f\in Z_{p-1,0}$.
24$_p$. $j_{x^3y,y^{3p+1}}f = y(x^3+x^2y^p) \Rightarrow f\in Z_{p-1,r}\ (r>0)$.

The series W.
25. $j_4f(x,y) = x^4 \Rightarrow j_{x^4,y^4}f = x^4 \Rightarrow \mathbf{26}_1$.

In Theorems **26–35**, $k\geqq 1$.
26$_k$. $j_{x^4,y^{4k}}f = x^4 \Rightarrow$ one of three cases:

$$\begin{aligned} j_{x^4,y^{4k+1}}f &\approx x^4+y^{4k+1} &&\models\!\Rightarrow \mathbf{27}_k;\\ j_{x^4,xy^{3k+1}}f &\approx x^4+xy^{3k+1} &&\models\!\Rightarrow \mathbf{28}_k;\\ j_{x^4,xy^{3k+1}}f &= x^4 &&\models\!\Rightarrow \mathbf{29}_k. \end{aligned}$$

27$_k$. $j_{x^4,y^{4k+1}}f = x^4+y^{4k+1} \Rightarrow f\in W_{12k}$.
28$_k$. $j_{x^4,xy^{3k+1}}f = x^4+xy^{3k+1} \Rightarrow f\in W_{12k+1}$.
29$_k$. $j_{x^4,xy^{3k+1}}f = x^4 \Rightarrow$ one of four cases:

$$\begin{aligned} j_{x^4,y^{4k+2}}f &\approx x^4+bx^2y^{2k+1}+y^{4k+2}, \quad b^2\neq 4 &&\models\!\Rightarrow \mathbf{30}_k;\\ &\approx x^4+x^2y^{2k+1} &&\models\!\Rightarrow \mathbf{31}_k;\\ &\approx (x^2+y^{2k+1})^2 &&\models\!\Rightarrow \mathbf{32}_k;\\ &= x^4 &&\models\!\Rightarrow \mathbf{33}_k. \end{aligned}$$

30$_k$. $j_{x^4,y^{4k+2}}f = x^4+bx^2y^{2k+1}+y^{4k+2}, \quad b^2\neq 4 \Rightarrow f\in W_{k,0}$
31$_k$. $j_{x^4,y^{4k+2}}f = x^4+x^2y^{2k+1} \quad \Rightarrow f\in W_{k,i}\ (i>0)$.
32$_k$. $j_{x^4,y^{4k+2}}f = (x^2+y^{2k+1})^2 \quad \Rightarrow f\in W_{k,i}\ (i>0)$.
33$_k$. $j_{x^4,y^{4k+2}}f = x^4 \Rightarrow$ one of three cases:

$$\begin{aligned} j_{x^4,xy^{3k+2}}f &\approx x^4+xy^{3k+2} &&\models\!\Rightarrow \mathbf{34}_k;\\ j_{x^4,y^{4k+3}}f &\approx x^4+y^{4k+3} &&\models\!\Rightarrow \mathbf{35}_k;\\ j_{x^4,y^{4k+3}}f &= x^4 &&\models\!\Rightarrow \mathbf{36}_{k+1}. \end{aligned}$$

34$_k$. $j_{x^4,xy^{3k+2}}f = x^4+xy^{3k+2} \Rightarrow f\in W_{12k+5}$.
35$_k$. $j_{x^4,y^{4k+3}}f = x^4+y^{4k+3} \Rightarrow f\in W_{12k+6}$.

In Theorems **36–46**, $k>1$.
The series X_k
36$_k$. $j_{x^4,y^{4k-1}}f = x^4 \Rightarrow$ one of five cases:

$$\begin{aligned} j_{x^4,y^{4k}}f &\approx x^4+bx^3y^k+ax^2y^{2k}+xy^{3k}, \quad \Delta\neq 0, \quad ab\neq 9 &&\models\!\Rightarrow \mathbf{37}_k;\\ &\approx x^2(x^2+axy^k+y^{2k}), \quad a^2\neq 4 &&\models\!\Rightarrow \mathbf{38}_k;\\ &\approx x^2(x+y^k)^2 &&\models\!\Rightarrow \mathbf{39}_k;\\ &\approx x^3(x+y^k) &&\models\!\Rightarrow \mathbf{40}_k;\\ &\approx x^4 &&\models\!\Rightarrow \mathbf{26}_k. \end{aligned}$$

37$_k$. $j_{x^4,y^{4k}}f = x^4+bx^3y^k+ax^2y^{2k}+xy^{3k}, \quad \Delta\neq 0 \Rightarrow f\in X_{k,0}$.
38$_k$. $j_{x^4,y^{4k}}f = x^2(x^2+axy^k+y^{2k}), \quad a^2\neq 4 \quad \Rightarrow f\in X_{k,p}\ (p>0)$.
39$_k$. $j_{x^4,y^{4k}}f = x^2(x^2+y^k)^2 \quad \Rightarrow f\in Y^k_{r,s}\ (1\leqq s\leqq r)$.
40$_k$. $j_{x^4,y^{4k}}f = (x+y^k)x^3 \quad \Rightarrow f\sim f_1f_2$, where

$$j_{x,y^k} f_1 \approx x + y^k, \quad j_{x^3,y^{3k}} f_2 = x^3 \models\!\Rightarrow \mathbf{41_k}.$$

In Theorems **41–44**, $i \geqq 0$, $p > 0$.

41_k. $j_{x^3,y^{3k}} f_2 = x^3 \Rightarrow$ one of five cases:

$$\begin{aligned} &f_2 \in E_{6(k+i)} &&\models\!\Rightarrow \mathbf{42_{k,i}};\\ &f_2 \in E_{6(k+i)+1} &&\models\!\Rightarrow \mathbf{43_{k,i}};\\ &f_2 \in E_{6(k+i)+2} &&\models\!\Rightarrow \mathbf{44_{k,i}};\\ &f_2 \in J_{k+i+1,0} &&\models\!\Rightarrow \mathbf{45_{k,i+1}};\\ &f_2 \in J_{k+i+1,p} &&\models\!\Rightarrow \mathbf{46_{k,i+1,p}}.\end{aligned}$$

In Theorems **42–46**, $f(x, y) = f_1 f_2$, where $j_{x,y^k} f_1 \approx x + y^k$ and $j_{x^3,y^{3k}} f_2 = x^3$.

$42_{k,i}$. $f_2 \in E_{6(k+i)} \quad \Rightarrow f \in Z^k_{12k+6i-1}$.

$43_{k,i}$. $f_2 \in E_{6(k+i)+1} \Rightarrow f \in Z^k_{12k+6i}$.

$44_{k,i}$. $f_2 \in E_{6(k+i)+2} \Rightarrow f \in Z^k_{12k+6i+1}$.

In Theorems **45–46**, $i \geqq 1$, $p > 0$.

$45_{k,i}$. $f_2 \in J_{k+i,0} \quad \Rightarrow f \in Z^k_{i,0}$.

$46_{k,i,p}$. $f_2 \in J_{k+i,p} \Rightarrow f \in Z^k_{i,p}$.

47. $j_4 f = 0 \Rightarrow$ one of two cases:

$$\begin{aligned} &j_5 f \approx x^4y + ax^3y^2 + bx^2y^3 + xy^4, \quad \Delta \neq 0, \quad ab \neq 9 \models\!\Rightarrow \mathbf{48};\\ &j_5 f \text{ is degenerate} \models\!\Rightarrow \mathbf{49}.\end{aligned}$$

48. $j_5 f = x^4y + ax^3y^2 + bx^2y^3 + xy^4$, $\Delta \neq 0 \Rightarrow f \in N_{16}$, i.e. $f \sim x^4y + ax^3y^2 + bx^2y^3 + xy^4 + cx^3y^3$, $\Delta \neq 0$, $ab \neq 9$; $\mu(f) = 16$, $m(f) = 3$, $c(f) = 12$.

49. $j_5 f$ is degenerate $\Rightarrow \mu(f) > 16$, $m(f) > 2$, $c(f) > 12$.

Singularities of corank 3

In Theorems **50–104**, $f \in \mathbb{C}[[x, y, z]]$.

50. $j_2 f(x, y, z) = 0 \Rightarrow$ one of ten cases:

$$\begin{aligned} j_3 f &\approx x^3 + y^3 + z^3 + axyz, \quad a^3 + 27 \neq 0 &&\models\!\Rightarrow \mathbf{51};\\ &\approx x^3 + y^3 + xyz &&\models\!\Rightarrow \mathbf{52} \text{ (series } \boldsymbol{P});\\ &\approx x^3 + xyz &&\models\!\Rightarrow \mathbf{54} \text{ (series } \boldsymbol{R});\\ &\approx xyz &&\models\!\Rightarrow \mathbf{56} \text{ (series } \boldsymbol{T});\\ &\approx x^3 + yz^2 &&\models\!\Rightarrow \mathbf{58} \text{ (series } \boldsymbol{Q});\\ &\approx x^2z + yz^2 &&\models\!\Rightarrow \mathbf{66} \text{ (series } \boldsymbol{S});\\ &\approx x^3 + xz^2 &&\models\!\Rightarrow \mathbf{82} \text{ (series } \boldsymbol{U});\\ &\approx x^2y &&\models\!\Rightarrow \mathbf{97} \text{ (class } \boldsymbol{V});\\ &\approx x^3 &&\models\!\Rightarrow \mathbf{103};\\ &= 0 &&\models\!\Rightarrow \mathbf{104}.\end{aligned}$$

*Series **T***

51. $j_3 f = x^3 + y^3 + z^3 + axyz$, $a^3 + 27 \neq 0 \Rightarrow f \in P_8 = T_{3,3,3}$.

52. $j_3 f = x^3 + y^3 + xyz \Rightarrow f \sim x^3 + y^3 + xyz + \alpha(z)$, $j_3(\alpha) = 0 \models\!\Rightarrow \mathbf{53}$.

53. $f = x^3 + y^3 + xyz + \alpha(z)$, $j_3(\alpha) = 0 \Rightarrow f \in P_{p+5} = T_{3,3,3+p}$ $(p > 3)$.

54. $j_3 f = x^3 + xyz \Rightarrow f \sim x^3 + xyz + \alpha(y) + \beta(z)$, $j_3(\alpha, \beta) = 0 \models\!\Rightarrow \mathbf{55}$.

55. $f = x^3 + xyz + \alpha(y) + \beta(z)$, $j_3(\alpha, \beta) = 0 \Rightarrow f \in R_{p,q} = T_{3,p,q}$ $(q \geqq p > 3)$.

56. $j_3 f = xyz \Rightarrow f \sim xyz + \alpha(x) + \beta(y) + \gamma(z)$, $j_3(\alpha, \beta, \gamma) = 0 \models\!\Rightarrow \mathbf{57}$.

57. $f = xyz + \alpha(x) + \beta(y) + \gamma(z)$, $j_3(\alpha, \beta, \gamma) = 0 \Rightarrow f \in T_{p,q,r}$ $(r \geqq q \geqq p > 3)$.

*Series **Q**.* In Theorems **58–65**,

$$\varphi = x^3 + yz^2, \; j^*_\lambda = j_{yz^2,x^3,\lambda} \; (\lambda \text{ is a monomial}).$$

58. $j_3 f = \varphi \Rightarrow f = \varphi + \alpha(y) + x\beta(y)$, $j_3(\alpha, x\beta) = 0 \models\!\Rightarrow \mathbf{59_1}$.

In Theorems **59–62**, $k \geqq 1$.

59_k. $f = \varphi + \alpha(y) + x\beta(y)$, $j^*_{y^{3k}} f = \varphi \Rightarrow$ one of four cases:

$$\begin{aligned} &j^*_{y^{3k+1}} f \approx \varphi + y^{3k+1} &&\models\!\Rightarrow \mathbf{60_k};\\ &j^*_{xy^{2k+1}} f \approx \varphi + xy^{2k+1} &&\models\!\Rightarrow \mathbf{61_k};\\ &j^*_{y^{3k+2}} f \approx \varphi + y^{3k+2} &&\models\!\Rightarrow \mathbf{62_k};\\ &j^*_{y^{3k+2}} f = \varphi &&\models\!\Rightarrow \mathbf{63_{k+1}}.\end{aligned}$$

60_k. $j^*_{y^{3k+1}} f \;= \varphi + y^{3k+1} \quad \Rightarrow f \in Q_{6k+4}$.

61_k. $j^*_{xy^{2k+1}} f = \varphi + xy^{2k+1} \Rightarrow f \in Q_{6k+5}$.

62_k. $j^*_{y^{3k+2}} f \;= \varphi + y^{3k+2} \quad \Rightarrow f \in Q_{6k+6}$.

In Theorems **63–65**, $k > 1$.

63_k. $f = \varphi + \alpha(y) + \beta(y)$, $j^*_{y^{3k-1}} f = \varphi \Rightarrow$ one of three cases:

$$\begin{aligned} j^*_{y^{3k}} f &\approx \varphi + ax^2y^k + xy^{2k}, \quad a^2 \neq 4 &&\models\!\Rightarrow \mathbf{64_k};\\ &\approx \varphi + x^2y^k &&\models\!\Rightarrow \mathbf{65_k};\\ &= \varphi &&\models\!\Rightarrow \mathbf{59_k}.\end{aligned}$$

64_k. $j^*_{y^{3k}} f = \varphi + ax^2y^k + xy^{2k}$, $\quad a^2 \neq 4 \Rightarrow f \in Q_{k,0}$.

65_k. $j^*_{y^{3k}} f = \varphi + x^2y^k \quad \Rightarrow f \in Q_{k,i}$ $(i > 0)$.

*Series **S**.* In Theorems **68–81**,

$$\varphi = x^2z + yz^2, \; j^*_\lambda = j_{x^2z,yz^2,\lambda} \; (\lambda \text{ is a monomial}).$$

66. $j_3 f = \varphi \Rightarrow f \sim \varphi + \alpha(y) + x\beta(y) + z\gamma(y)$, $j_3(\alpha, x\beta, z\gamma) = 0 \models\!\Rightarrow \mathbf{67_1}$.

In Theorems **67–76**, $k \geqq 1$.

67_k. $f = \varphi + \alpha(y) + x\beta(y) + z\gamma(y)$, $j^*_{y^{4k-1}} f = \varphi \Rightarrow$ one of three cases:

$$\begin{aligned} &j^*_{y^{4k}} f \approx \varphi + y^{4k} &&\models\Rightarrow \mathbf{68}_k;\\ &j^*_{xy^{3k}} f \approx \varphi + xy^{3k} &&\models\Rightarrow \mathbf{69}_k;\\ &j^*_{xy^{3k}} f = \varphi &&\models\Rightarrow \mathbf{70}_k. \end{aligned}$$

68$_k$. $j^*_{y^{4k}} f = \varphi + y^{4k}x \Rightarrow f \in S_{12k-1}$.
69$_k$. $j^*_{xy^{3k}} f = \varphi + xy^{3k} \Rightarrow f \in S_{12k}$.
70$_k$. $f = \varphi + \alpha(y) + x\beta(y) + z\gamma(y)$, $j^*_{xy^{3k}} f = \varphi \Rightarrow$ one of four cases:

$$\begin{aligned} j^*_{y^{4k+1}} f &\approx \varphi + y^{4k+1} + bzy^{2k+1}, \quad b^2 \neq 4 &&\models\Rightarrow \mathbf{71}_k;\\ &\approx \varphi + x^2y^{2k} &&\models\Rightarrow \mathbf{72}_k;\\ &\approx \varphi + zy^{2k+1} &&\models\Rightarrow \mathbf{73}_k;\\ &= \varphi &&\models\Rightarrow \mathbf{74}_k. \end{aligned}$$

71$_k$. $j^*_{y^{4k+1}} f = \varphi + y^{4k+1} + bzy^{2k+1}$, $b^2 \neq 4 \Rightarrow f \in S_{k,0}$.
72$_k$. $j^*_{y^{4k+1}} f = \varphi + x^2y^{2k} \Rightarrow f \in S_{k,i}$ $(i > 0)$.
73$_k$. $j^*_{y^{4k+1}} f = \varphi + zy^{2k+1} \Rightarrow f \in S_{k,i} (i > 0)$.
74$_k$. $f = \varphi + \alpha(y) + x\beta(y) + z\gamma(y)$, $j^*_{y^{4k+1}} f = \varphi \Rightarrow$ one of three cases:

$$\begin{aligned} &j^*_{xy^{3k+1}} f = \varphi + xy^{3k+1} &&\models\Rightarrow \mathbf{75}_k;\\ &j^*_{y^{4k+2}} f = \varphi + y^{4k+2} &&\models\Rightarrow \mathbf{76}_k;\\ &j^*_{y^{4k+2}} f = \varphi &&\models\Rightarrow \mathbf{77}_{k+1}. \end{aligned}$$

75$_k$. $j^*_{xy^{3k+1}} f = \varphi + xy^{3k+1} \Rightarrow f \in S_{12k+4}$.
76$_k$. $j^*_{y^{4k+2}} f = \varphi + y^{4k+2} \Rightarrow f \in S_{12k+5}$.

In Theorems **77–81**, $k > 1$.
77$_k$. $f = \varphi + \alpha(y) + x\beta(y) + z\gamma(y)$, $j^*_{y^{4k-2}} f = \varphi \Rightarrow$ one of five cases:

$$\begin{aligned} j^*_{y^{4k-1}} f &\approx \varphi + ax^2y^{2k-1} + bxy^kz + xy^{3k-1}, \quad \Delta \neq 0 &&\models\Rightarrow \mathbf{78}_k;\\ &\approx \varphi + xy^kz + ax^3y^{k-1}, \quad a^2 \neq a &&\models\Rightarrow \mathbf{79}_k;\\ &\approx \varphi + x^3y^{k-1} &&\models\Rightarrow \mathbf{80}_k;\\ &\approx \varphi + xy^kz &&\models\Rightarrow \mathbf{81}_k;\\ &= \varphi &&\models\Rightarrow \mathbf{67}_k. \end{aligned}$$

78$_k$. $j^*_{y^{4k-1}} f = \varphi + ax^2y^{2k-1} + bxy^kz + xy^{3k-1}$, $\Delta \neq 0 \Rightarrow f \in S^*_{k,0}$; $\mu(f) = 12k-4$, $m(f) \geqq 3k-2$, $c(S^*_{k,0}) = 9k-3$.
79$_k$. $j^*_{y^{4k-1}} f = \varphi + xy^kz + ax^3y^{k-1}$, $a^2 \neq a \Rightarrow f \in \boldsymbol{SP}_k$; $\mu(f) \geqq 12k-3$, $m(f) \geqq 3k-2$, $c(\boldsymbol{SP}_k) = 9k-2$.
80$_k$. $j^*_{y^{4k-1}} f = \varphi + x^3y^{k-1} \Rightarrow f \in \boldsymbol{SQ}_k$; $\mu(f) \geqq 12k-2$, $m(f) \geqq 3k-2$, $c(\boldsymbol{SQ}_k) = 9k-1$.
81$_k$. $j^*_{y^{4k-1}} f = \varphi + xy^kz \Rightarrow f \in \boldsymbol{SR}_k$; $\mu(f) \geqq 12k-2$, $m(f) \geqq 3k-2$, $c(\boldsymbol{SR}_k) = 9k-1$.

The *series U*.
In Theorems **82–89**,

$$\varphi = x^3, + xz^2, j^*_\lambda = j_{x^3,z^3,\lambda} \ (\lambda \text{ is a monomial}).$$

82. $j_3 f = \varphi \Rightarrow f \sim \varphi + \alpha(y) + x\beta(y) + z\gamma(y) + x^2\delta(y)$, $j_3(\alpha, x\beta, z\gamma, x^2\delta) = 0 \Rightarrow \mathbf{83}_1$.

In Theorems **83–89**, $k \geqq 1$.
83$_k$. $f = \varphi + \alpha(y) + x\beta(y) + z\gamma(y) + x^2\delta(y)$, $j^*_{y^{3k}} f = \varphi \Rightarrow$ one of two cases:

$$\begin{aligned} j^*_{y^{3k+1}} f &\approx \varphi + y^{3k+1} &&\models\Rightarrow \mathbf{84}_k,\\ &= \varphi &&\models\Rightarrow \mathbf{85}_k. \end{aligned}$$

84$_k$. $j^*_{y^{3k+1}} f = \varphi + y^{3k+1} \Rightarrow f \in U_{12k}$.
85$_k$. $f = \varphi + \alpha(y) + x\beta(y) + z\gamma(y) + x^2\delta(y)$, $j^*_{y^{3k+1}} f = \varphi \Rightarrow$ one of three cases:

$$\begin{aligned} j^*_{xy^{2k+1}} f &\approx \varphi + xy^{2k+1} + czy^{2k+1}, \quad c(c^2+1) \neq 0 &&\models\Rightarrow \mathbf{86}_k;\\ &\approx \varphi + xy^{2k+1} &&\models\Rightarrow \mathbf{87}_k;\\ &= \varphi &&\models\Rightarrow \mathbf{88}_k. \end{aligned}$$

86$_k$. $j^*_{xy^{2k+1}} f = \varphi + xy^{2k+1} + czy^{2k+1}$, $c(c^2+1) \neq 0 \Rightarrow f \in U_{k,0}$.
87$_k$. $j^*_{xy^{2k+1}} f = \varphi + xy^{2k+1} \Rightarrow f \in U_{k,p}$ $(p > 0)$.
88$_k$. $f = \varphi + \alpha(y) + x\beta(y) + z\gamma(y) + x^2\delta(y)$, $j^*_{xy^{2k+1}} f = \varphi \Rightarrow$ one of two cases:

$$\begin{aligned} j^*_{y^{3k+2}} f &\approx \varphi + y^{3k+2} &&\models\Rightarrow \mathbf{89}_k;\\ &= \varphi &&\models\Rightarrow \mathbf{90}_{k+1}. \end{aligned}$$

89$_k$. $j^*_{y^{3k+2}} f = \varphi + y^{3k+2} \Rightarrow f \in U_{12k+4}$.

In Theorems **90–96**, $k \geqq 2$;

$$\varphi = x^2z + xz^2, j^*_\lambda = j_{x^3,z^3,\lambda} \ (\lambda \text{ is a monomial}).$$

90$_k$. $f = x^3 + xz^2 + \alpha(y) + x\beta(y) + z\gamma(y) + x^2\delta(y)$, $j^*_{y^{3k-1}} f = x^3 + xz^2 \Rightarrow$ one of seven cases:

$$\begin{aligned} j^*_{y^{3k}} f &\approx \varphi + ax^2y^k + bxy^kz + y^kz^2 + cxy^{2k}, \quad \Delta \neq 0 &&\models\Rightarrow \mathbf{91}_k;\\ &\approx \varphi + ax^2y^k + bxy^kz + y^kz^2, \quad 4a \neq b^2,\ a(a+1-b) \neq 0 &&\models\Rightarrow \mathbf{92}_k;\\ &\approx x^3 + ax^2z + xz^2 + y^kz^2, \quad a^2 \neq 4 &&\models\Rightarrow \mathbf{93}_k;\\ &\approx \varphi + x^2y^k + axy^kz, \quad a^2 \neq a &&\models\Rightarrow \mathbf{94}_k;\\ &\approx \varphi + x^2y^k &&\models\Rightarrow \mathbf{95}_k;\\ &\approx \varphi + xy^kz &&\models\Rightarrow \mathbf{96}_k;\\ &= \varphi &&\models\Rightarrow \mathbf{83}_k. \end{aligned}$$

91$_k$. $j^*_{y^{3k}} f = \varphi + ax^2y^k + bxy^kz + y^kz^2 + cxy^{2k}$, $\Delta \neq 0 \Rightarrow f \in U^*_{k,0}$; $\mu(f) = 12k-4$, $m(f) \geqq 4k-3$, $c(U^*_{k,0}) = 8k-2$.
92$_k$. $j^*_{y^{3k}} f = \varphi + ax^2y^k + bxy^kz + y^kz^2$, $4a \neq b$, $a(a+1-b) \neq 0 \Rightarrow f \in \boldsymbol{UP}_k$; $\mu(f) \geqq 12k-3$, $m(f) \geqq 4k-3$, $c(\boldsymbol{UP}_k) = 8k-1$.
93$_k$. $j^*_{y^{3k}} f = x^3 + ax^2z + xz^2 + y^kz^2$, $a^2 \neq 4 \Rightarrow f \in \boldsymbol{UQ}_k$; $\mu(f) \geqq 12k-2$, $m(f) \geqq 4k-3$, $c(\boldsymbol{UQ}_k) = 8k$.

94$_k$. $j^*_{y^{3k}}f = \varphi + x^2y^k + axy^kz$, $a^2 \neq a \Rightarrow f \in UR_k$; $\mu(f) \geqq 12k-2$, $m(f) \geqq 4k-3$, $c(UR_k) = 8k$.

95$_k$. $j^*_{y^{3k}}f = \varphi + x^2y^k \Rightarrow f \in US_k$, $\mu(f) \geqq 12k-1$, $m(f) \geqq 4k-3$, $c(US_k) = 8k+1$.

96$_k$. $j^*_{y^{3k}}f = \varphi + xy^kz \Rightarrow f \in UT_k$, $\mu(f) \geqq 12k-1$, $m(f) \geqq 4k-3$, $c(UT_k) = 8k+1$.

The *Class V*.

97. $j_3f(x, y, z) = x^2y \Rightarrow f \sim x^2y + \alpha(y, z) + x\beta(z) \models$ **98**.

In Theorems **98** and **102**, φ is one of the following 10 polynomials

$$z^4 + z^3y,\ z^3y + z^2y^2,\ z^2y^2 + zy^3,\ z^4 + z^2y^2,\ z^4,\ z^3y,\ z^2y^2,\ zy^3,\ y^4,\ 0.$$

98. $f = x^2y + \alpha(y, x) + x\beta(z)$, $j_3f = x^2y \Rightarrow$ one of four cases:

$$\begin{aligned} j_{x^2y,y^4,z^4}f &\approx x^2y + z^4 + az^3y + bz^2y^2 + zy^3,\ \Delta \neq 0,\ ab \neq 9 \models \mathbf{99};\\ &\approx x^2y + z^4 + bz^3y + z^2y^2,\ b^2 \neq 4 \models \mathbf{100};\\ &\approx x^2y + z^3y + az^2y^2 + y^4,\ 4a^3 + 27 \neq 0 \models \mathbf{101};\\ &\approx x^2y + \varphi \models \mathbf{102}.\end{aligned}$$

99. $j_{x^2y,y^4,z^4}f = x^2y + z^4 + az^3y + bz^2y^2 + zy^3$, $\Delta \neq 0 \Rightarrow f \in V_{1,0}$.

100. $j_{x^2y,y^4,z^4}f = x^2y + z^4 + bz^3y + z^2y^2$, $b^2 \neq 4 \Rightarrow f \in V_{1,p}$ $p > 0$.

101. $j_{x^2y,y^4,z^4}f = x^2y + z^3y + az^2y^2 + y^4$, $4a^3 + 27 \neq 0 \Rightarrow f \in V^{\#}_{1,p}$ $p > 0$.

102. $j_{x^2y,y^4,z^4}f = x^2y + \varphi \Rightarrow \mu(f) \geqq 17$, $m(f) \geqq 3$, $c(f) \geqq 13$.

103. $j_3f(x, y, z) = x^3 \Rightarrow \mu(f) \geqq 18$, $m(f) \geqq 4$, $c(f) \geqq 13$.

104. $j_3f(x, y, z) = 0 \Rightarrow \mu(f) \geqq 27$, $m(f) \geqq 10$, $c(f) \geqq 16$.

105. $\operatorname{corank} f > 3 \Rightarrow \mu(f) > 16$, $m(f) \geqq 5$, $c(f) \geqq 10$.

16.3 Proofs

Theorems **1**, **17** and **25** are obvious. Theorems **6**, **18**, **26**, **33**, **59**, **67**, **74**, **83** and **88** are proved by the Newton ruler method (see Section 11.2; there is a more detailed proof in [17]). Theorems **3**, **10**, **13**, **22**, **29**, **36**, **47**, **50**, **52**, **54**, **56**, **58**, **63**, **66**, **70**, **77**, **82**, **85**, **90**, **97** and **98** establish normal forms for quasihomogeneous singularities; the roots technique for finding these forms is described in Chapter 13 (there are more detailed proofs in [17]).

The method of Chapter 13 reduces the proofs of these theorems to the following problems of geometrical classification:

Theorem No.	Series	Geometric problem
3	**D**	Linear classification of 3-forms in $\mathbb{C}^2$
10, 22	**J, Z**	Affine classification of triples of points in $\mathbb{C}^1$
13	**X**	Linear classification of 4-forms in $\mathbb{C}^2$
29	**W**	Linear classification of pairs of points in $\mathbb{C}^1$
36	**X**	Affine classification of quadruples of points in $\mathbb{C}^1$
47	**N**	Linear classification of 5-forms in $\mathbb{C}^2$
50	**P**	Linear classification of 3-forms in $\mathbb{C}^3$
63	**Q**	Affine classification of cubic curves in $\mathbb{C}^2$ with a finite cusp
70	**S**	Affine classification of reducible cubic curves in $\mathbb{C}^2$ having at least two finite double points
77	**S***	Affine classification of cubic curves in $\mathbb{C}^2$ having a simple infinite tangent
85	**U**	Affine classification of centrally-symmetric cubic curves in $\mathbb{C}^2$ with precisely three points at infinity
90	**U***	Affine classification of cubic curves in $\mathbb{C}^2$ with precisely three points at infinity
98	**V**	Affine classification of polynomials of degree $\leqq 4$ in one variable in $\mathbb{C}^1$

Theorem **50** describes the stratification of the 10-dimensional space of cubic forms depicted in Fig. 54 according to singularities (see [147]).

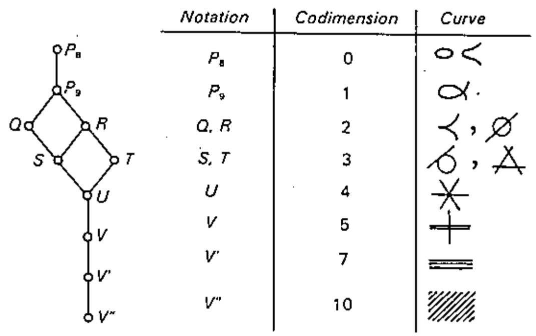


Fig. 54.

Theorems **2, 4, 5, 7, 8, 9, 11, 14, 19, 20, 21, 23, 27, 28, 30, 34, 35, 37, 48, 51, 60, 61, 62, 64, 68, 69, 71, 75, 76, 78, 84, 86, 89, 91** and **99** are Corollaries of the theorem on the normal form of semiquasihomogeneous singularities (Section 12.6). For detailed proofs see [17].

Theorems $\mathbf{12_2}$, **15, 16, 53, 55** and **57** are proved by the crossword method, details in [17]. Theorems **49** and **102–105** are also proved by the methods of Chapter 12.

The proofs of Theorems $\mathbf{12_k}$ $(k > 2)$, **24, 31, 32, 38–46, 65, 72, 73, 79, 80, 81, 87, 92–96, 100** and **101** are based on the spectral sequence (see Chapter 14; the detailed computations are in [18]).

Proof of the theorem on the classification of simple singularities

(1°) *Every singularity either belongs to the list of simple singularities of Section* 15.1.0 *or is adjacent to one of the classes* P_8, X_9, J_{10} (these three classes we shall call the *confining classes for the simple singularities*).

The proof follows from Theorems **1–5**, $\mathbf{6_1}$–$\mathbf{11_1}$, **13, 14, 50** and **51**.

Moreover Theorems $\mathbf{10_1}$, **13** and **50** are only used in part: only the first case is needed in each of them.

(2°) *The modality of the singularities of each of the three confining classes is not less than* 1.

This follows from the fact that their inner modality is equal to 1. The geometrical meaning of this proposition is the following: a cubic curve in $\mathbb{C}P^2$, a set of four lines passing through zero in $\mathbb{C}^2$ and three parabolas in $\mathbb{C}^2$ touching at a point all have moduli (under the actions of the projective group, the linear group and the group of 2-jets of diffeomorphisms respectively). For example, the modulus of a set of four lines is its cross-ratio.

(3°) *The singularities of the list are not adjacent to the confining ones.* P_8 has corank 3 and X_9 is equivalent to a function of two variables with zero 3-jet. The functions of two variables in the list have corank $\leqq 2$ and a nonzero 3-jet and therefore adjacencies to P_8 and X_9 are excluded. Adjacencies to J_{10} are excluded by the proof of Theorem $\mathbf{6_1}$.

From (1°), (2°) and (3°) it follows that these singularities in the list and only these are simple.

Proof of the theorem on the classification of unimodal singularities

(1°) *Each non-simple singularity either belongs to the list of unimodal singularities of Section* 15.1.1 *or is adjacent to one of the following ten singularity classes:*

$J_{3,0}$, $W_{1,0}$, $Z_{1,0}$, N, $Q_{2,0}$, $S_{1,0}$, $U_{1,0}$, V and O (we shall call these ten classes the *confining classes for the unimodal singularities*).

The proof follows from Theorems **1–5**, $\mathbf{6_{1,2}}$–$\mathbf{9_{1,2}}$, $\mathbf{10_2}$, $\mathbf{11_2}$, **13–17**, $\mathbf{18_1}$–$\mathbf{21_1}$, $\mathbf{22_2}$, $\mathbf{23_2}$, **25**, $\mathbf{26_1}$–$\mathbf{30_1}$, $\mathbf{36_1}$, $\mathbf{37_1}$, **47, 48, 50–58**, $\mathbf{59_1}$–$\mathbf{62_1}$, $\mathbf{63_2}$, $\mathbf{64_2}$, **66**, $\mathbf{67_1}$–$\mathbf{71_1}$, **82**, $\mathbf{83_1}$–$\mathbf{86_1}$, **97, 98** and **105**. Theorems **10, 22, 29, 36, 47, 63, 79, 85** and **98** are used only in part (the first case in each of them is all that is required).

(2°) *The modality of each singularity of the list is not less than* 1 *while the modality of each singularity of any of the ten confining classes is greater than* 1.

The first follows from the fact that all the singularities of the list are adjacent to P_8, X_9 or J_{10} while the second follows from the fact that the inner modality of the confining singularities is greater than 1.

(3°) *The singularities of the list are not adjacent to the confining singularities.*

The quadratic forms of the singularities of all the confining classes have not less than two positive squares (see [11]). The positive index of inertia of the quadratic form of a singularity is semicontinuous (see, for example, [169]). The positive index of inertia of the quadratic forms of the $T_{p,q,r}$ singularities is less than two (see [67]). Consequently the adjacency of the $T_{p,q,r}$ singularities (whether parabolic or hyperbolic) to the confining singularities is not possible. Thus all the $T_{p,q,r}$ singularities are unimodal.

The confining singularities of corank 2 have $\mu \geqq 15$, while for all the exceptional singularities $\mu \leqq 14$. Consequently the adjacency of the exceptional singularities to the confining singularities of corank 2 is not possible.

Of the exceptional singularities only those of corank 3 could be adjacent to the confining singularities of corank greater than two. But the confining singularities of corank greater than two have $\mu \geqq 14$, while the exceptional singularities have $\mu \leqq 12$. Therefore such adjacencies also are not possible.

From (1°), (2°), (3°) it follows that all the singularities of the list, and only these, are unimodal.

Proof of the theorem on the classification of the bimodal singularities

(1°) *Each non-simple and non-unimodal singularity either belongs to the list of bimodal singularities of Section* 15.1.2 *or is adjacent to one of the ten classes:* $J_{4,0}$, $X_{2,0}$, $Z_{2,0}$, N, $Q_{3,0}$, $S^{*}_{2,0}$, $U^{*}_{2,0}$, V, O (*we shall call these ten classes the confining classes for the bimodal classes*).

The proof is obtained by supplementing the theorems quoted in (1°) of the preceding proof with Theorems $\mathbf{6_3}$–$\mathbf{11_3}$, $\mathbf{12_2}$, $\mathbf{18_2}$–$\mathbf{21_2}$, $\mathbf{22_3}$, $\mathbf{23_3}$ $\mathbf{24_2}$, $\mathbf{31_1}$–$\mathbf{35_1}$, $\mathbf{36_2}$, $\mathbf{37_2}$, $\mathbf{59_2}$–$\mathbf{62_2}$, $\mathbf{63_3}$, $\mathbf{64_3}$, $\mathbf{65_2}$, $\mathbf{72_1}$–$\mathbf{76_1}$, $\mathbf{77_2}$, $\mathbf{78_2}$, $\mathbf{87_1}$, $\mathbf{89_1}$, $\mathbf{90_2}$, $\mathbf{91_2}$, **99**. Moreover Theorems $\mathbf{10_3}$, $\mathbf{22_3}$, $\mathbf{36_2}$, $\mathbf{63_3}$, $\mathbf{77_2}$, $\mathbf{90_2}$ are not used in full (only the first case in each of them is required).

(2°) *The modality of each singularity of the list is not less than 2, while the modality of each singularity of any of the ten confining classes is greater than 2.*

The first follows from the fact that all the singularities of the list are adjacent to the singularities of the ten classes of quasihomogeneous singularities confining the unimodal singularities while for the singularities of these classes the inner modality is not less than 2.

The second follows from the fact that the inner modality of the quasihomogeneous singularities confining the bimodal ones is greater than 2.

(3°) *The singularities of the list are not adjacent to the confining singularities.*

The quadratic forms of all the singularities of the list have exactly two positive squares in the normal form and are nondegenerate (see [11], [66]–[68]). The quadratic forms of the confining singularities, with the exception of N, V, and O, have two positive and two zero squares. The quadratic form of a deformed singularity is isomorphic to the restriction of the form of the original singularity to a subspace. This excludes all adjacencies of the singularities of the list to the confining ones, with the exception of N, V and O.

Adjacencies to O are excluded by the semicontinuity of the corank, while adjacencies to V are excluded by the stratification of cubic forms (Theorem **50**). Adjacencies to N of germs of the list having corank 2 are excluded by the semicontinuity of the order of a function at zero.

(4°) *Adjacencies to N of singularities of the list of corank 3 are not possible. Moreover, functions of three variables with the reduced 3-jets cannot be adjacent to the class N.*

In fact let $f = f_3 + f_4 + \dots$ be the decomposition into a Taylor series with reduced (that is, not having multiple factors) cubic form f_3. Let $\varphi = \varphi_2 + \varphi_3 + \dots$ be a small increment. If $f + \varphi$ is of class N then there exists a function u with noncritical point at zero such that $f + \varphi = u^2 \bmod m^5$. Let $u = u_1 + u_2 + u_3 \bmod \mathbf{m}^4$ be the first terms of the Taylor series for u. Then

$$\varphi_2 = u_1^2, \quad f_3 + \varphi_3 = 2u_1u_2, \quad f_4 + \varphi_4 = 2u_1u_3 + u_2^2.$$

This system of equations for u must be solvable for some sufficiently small φ, with $u_1 \neq 0$. Consequently f_3 decomposes into the product of a linear and a quadratic factor, $f_3 = F_1F_2$, such that the zeros of u_1 are close to the zeros of F_1, while those of u_2 are close to those of F_2: $u_1 = \varepsilon(F_1 + \alpha_1)$, $u_2 = (F_2 + \beta_2)/2\varepsilon$, where the number ε, the linear form α_1 and the quadratic form β_2 are small. Now from the last equation we conclude that there exist arbitrary small ε, φ_4, α_1 and β_2 such that $4\varepsilon^2(f_4 + \varphi_4) - (F_2 + \beta_2)^2$ is divisible by $F_1 + \alpha_1$. Accordingly F_2 is divisible by F_1, contrary to what we have been told about f_3.

Therefore the singularities of the classes $\boldsymbol{P}$, $\boldsymbol{Q}$, $\boldsymbol{R}$, $\boldsymbol{S}$, $\boldsymbol{T}$, $\boldsymbol{U}$ (in particular all the singularities of the list of corank 3) are not adjacent to N.

With this the proof of assertion (3°) is complete. From (1°), (2°), (3°) it follows that all the singularities of the list and only these are bimodal.